

STABLE SYSTEMS OF COMPETING LÉVY PARTICLES

ANDREY SARANTSEV

ABSTRACT. Take a finite system of Lévy particles on the real line. Each particle moves as a Lévy process according to its current rank relative to other particles. We find a natural sufficient condition for stability, when all particles move together, as opposed to eventually splitting into two or more groups. This generalizes a known condition for Brownian particles, where the average drift for a few bottom-ranked particles must be greater than the average drift for all particles.

1. INTRODUCTION

1.1. Definition of systems of competing Lévy particles. Assume the usual setting: a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbf{P})$, with the filtration satisfying the usual conditions. Fix $N \geq 2$, the number of particles. Take a Lévy process $L = (L(t), t \geq 0)$ in \mathbb{R}^N which makes a.s. finitely many jumps over any finite time interval. This process can be characterized by a triple (g, A, Λ) , where $g \in \mathbb{R}^N$ is a *drift vector*, A is an $N \times N$ positive definite symmetric *covariance matrix*, and Λ is a finite *Lévy measure* on \mathbb{R}^N . The process L behaves as an N -dimensional Brownian motion with drift vector g and covariance matrix A , except that it makes jumps. The times of jumps form a Poisson point process on the half-line $\mathbb{R}_+ := [0, \infty)$ with intensity $\lambda_0 = \Lambda(\mathbb{R}^N)$. Each jump has an independent displacement distributed according to the normalized measure $\lambda_0^{-1} \Lambda(\cdot)$ on \mathbb{R}^N . Now, consider an \mathbb{R}^N -valued continuous adapted process

$$X = (X(t), t \geq 0), \quad X(t) = (X_1(t), \dots, X_N(t)),$$

and rank components at every time $t \geq 0$: $X_{(1)}(t) \leq \dots \leq X_{(N)}(t)$. More rigorously, for a vector $x = (x_1, \dots, x_N) \in \mathbb{R}^N$, define the *ranking permutation* \mathbf{p}_x . This is a unique bijection $\{1, \dots, N\} \rightarrow \{1, \dots, N\}$ which has the following properties:

- (a) $x_{\mathbf{p}_x(i)}(t) \leq x_{\mathbf{p}_x(j)}(t)$ for $1 \leq i < j \leq N$;
- (b) if $1 \leq i < j \leq N$ and $x_{\mathbf{p}_x(i)}(t) = x_{\mathbf{p}_x(j)}(t)$, then $\mathbf{p}_x(i) < \mathbf{p}_x(j)$.

Assume the process X satisfies the following system of SDE:

$$(1) \quad dX_i(t) = \sum_{k=1}^N 1(\mathbf{p}_{X(t)}(k) = i) dL_k(t), \quad i = 1, \dots, N.$$

Then X is called a *system of competing Lévy particles, governed by a triple (g, A, Λ)* . Each component X_i is called the *i th named particle*. For each $k = 1, \dots, N$, the following process is called the *k th ranked particle*:

$$Y_k = (Y_k(t), t \geq 0), \quad Y_k(t) \equiv X_{\mathbf{p}_{X(t)}(k)}(t).$$

Date: October 17, 2016. Version 9.

2010 Mathematics Subject Classification. Primary 60J60, secondary 60J51, 60J75, 60H10, 60K35.

Key words and phrases. Lévy process, gap process, competing Brownian particles.

Remark 1. In this model, jumps of ranked particles are, in general, dependent. Assume we wish that ranked particles made independent jumps, with the jumps of the k th ranked particle governed by a finite Borel measure ν_k , $k = 1, \dots, N$. Then we need to take

$$(2) \quad \Lambda = \sum_{k=1}^N \Lambda_k, \quad \Lambda_k := \delta_0 \otimes \delta_0 \otimes \dots \otimes \delta_0 \otimes \nu_k \otimes \delta_0 \otimes \dots \otimes \delta_0.$$

Here, the Dirac point mass measure at $x \in \mathbb{R}$ is denoted by δ_x . For the measure Λ_k in (2), the multiple ν_k is on the k th place. Under condition (2), the k th ranked particle makes jumps with intensity $\lambda_k := \nu_k(\mathbb{R})$, and the displacement during each jump is distributed according to a probability measure $\lambda_k^{-1}\nu_k(\cdot)$. If, in addition,

$$(3) \quad A = \text{diag}(\sigma_1^2, \dots, \sigma_N^2),$$

then the components L_1, \dots, L_N of the process L are independent. In this case, k th ranked particle moves as a one-dimensional Lévy process $L_k = (L_k(t), t \geq 0)$ with drift coefficient g_k , diffusion coefficient σ_k^2 , and finite Lévy measure ν_k .

The following statement is proved in Section 3.

Lemma 1.1. *For every initial condition $X(0) = x \in \mathbb{R}^N$, the system (1) has a unique in law weak solution.*

1.2. Motivation and historical review. Similar systems without jumps are called *competing Brownian particles*. They were introduced (for the case of a diagonal A) in [2] for the purposes of financial modeling, and have been a subject of extensive research during the last decade: [3, 13, 14, 30, 31]. Further financial applications can be found in [7, 10, 11, 18, 20].

Systems of competing Brownian particles can also be viewed as a discrete analogue of a nonlinear diffusion process, governed by a McKean-Vlasov equation. In fact, as the number N of particles goes to infinity, the system converges to a nonlinear diffusion process in a certain sense, [8, 15, 16, 29, 36]. Also, these systems are scaling limits of exclusion processes, [19].

Systems of competing Lévy particles were introduced in [35] for the case (2) and (3), with $\nu_1 = \dots = \nu_N$. Systems of $N = 2$ competing Lévy particles (in the general case) were also studied in [33].

1.3. Notation. The dot product of two vectors $a = (a_1, \dots, a_d)$ and $b = (b_1, \dots, b_d)$ from \mathbb{R}^d is denoted by $a \cdot b = a_1 b_1 + \dots + a_d b_d$. For two finite Borel measures ρ' and ρ'' on \mathbb{R} , we say ρ' is *stochastically dominated* by ρ'' , and write $\rho' \preceq \rho''$, if $\rho'(\mathbb{R}) = \rho''(\mathbb{R})$, but $\rho'[z, \infty) \leq \rho''[z, \infty)$ for all $z \in \mathbb{R}$. The exponential distribution on \mathbb{R}_+ with rate λ (and mean λ^{-1}) is denoted by $\text{Exp}(\lambda)$. The *total variation* (TV) of a signed Borel measure ρ on \mathbb{R}_+^d is defined as the supremum over all Borel measurable functions $f : \mathbb{R}_+^d \rightarrow \mathbb{R}$:

$$\|\rho\|_{\text{TV}} = \sup_{|f| \leq 1} |(\rho, f)|, \quad (\rho, f) := \int_{\mathbb{R}_+^d} f(x) \rho(dx).$$

We say that random variables converge in TV norm if their distributions converge in this norm. Clearly, this convergence is stronger than weak convergence.

1.4. Stability of the system: main result. One important question is whether this system is *stable*, that is, all particles move together and do not diverge from each other. The *gap process* is a process $Z = (Z(t), t \geq 0)$ on \mathbb{R}_+^{N-1} defined by

$$(4) \quad Z(t) = (Z_1(t), \dots, Z_{N-1}(t)), \quad Z_k(t) = Y_{k+1}(t) - Y_k(t), \quad k = 1, \dots, N-1.$$

Definition 1. A Borel probability measure π on \mathbb{R}_+^{N-1} is called a *stationary gap distribution* for a system of competing Lévy particles from (1) if for every copy of this system with initial conditions satisfying $Z(0) \sim \pi$, we have: $Z(t) \sim \pi$ for every $t \geq 0$.

Definition 2. The system of competing Lévy particles is called *stable* if there exists a unique stationary gap distribution π , and regardless of the initial distribution $X(0)$, the law of $Z(t)$ converges to π as $t \rightarrow \infty$, in the total variation norm.

The topic of the current paper is to find when a system of competing Lévy particles is stable. For competing *Brownian* particles, this problem was solved in [3]. (This was proved for the case of diagonal matrix A , but the proof works for the general case.) The system is stable if and only if the average drift of the k bottom particles is strictly greater than the average drift of all particles, for each $k = 1, \dots, N-1$:

$$(5) \quad \frac{1}{k} (g_1 + \dots + g_k) > \frac{1}{N} (g_1 + \dots + g_N), \quad k = 1, \dots, N-1.$$

In other words, these two groups of particles travel together, and do not separate into two distinct “clouds”. For particles with jumps, it is natural to consider *effective drift coefficients*

$$(6) \quad m_k = g_k + \int_{\mathbb{R}^N} z_k \Lambda(dz), \quad k = 1, \dots, N,$$

instead of drift coefficients g_1, \dots, g_N . Indeed, the effective drift of the k th ranked particle is a combination of the *true drift* $g_k dt$, corresponding to the diffusion part of the process, and of the drift created by jumps. On a unit time interval, the system makes on average $\lambda_0 = \Lambda(\mathbb{R}^N)$ jumps. During each jump, the displacements of the k th ranked particle, for $k = 1, \dots, N$, form a vector distributed according to the probability measure $\lambda_0^{-1} \Lambda(\cdot)$. Therefore, the average displacement of the k th ranked particle is equal to

$$\int_{\mathbb{R}^N} z (\lambda_0^{-1} \Lambda(dz)).$$

Multiplying this average displacement by the average quantity of jumps, we get the term $\int_{\mathbb{R}^N} z_k \Lambda(dz)$. This is the “average shift created by jumps”. Adding it to the true drift coefficient g_k , we get (6). To make sense of (6), we assume that

$$(7) \quad \int_{\mathbb{R}^N} |z_k| \Lambda(dz) < \infty \text{ for } k = 1, \dots, N.$$

It is natural to guess that if a condition similar to (5) holds for competing Lévy particles, with effective drifts m_k instead of g_k , $k = 1, \dots, N$, then the system is stable. The main result of this article is that this is indeed true.

Theorem 1.2. *Under condition (7), assume the effective drifts m_1, \dots, m_N from (6) satisfy*

$$(8) \quad \frac{1}{k} (m_1 + \dots + m_k) > \frac{1}{N} (m_1 + \dots + m_N), \quad k = 1, \dots, N-1.$$

Then the system of competing Lévy particles from (1) is stable

For the case (2) in Remark 1, expression (6) and condition (7) take the form

$$m_k = g_k + \int_{\mathbb{R}} z \nu_k(dz), \quad \int_{\mathbb{R}} |z| \nu_k(dz) < \infty \text{ for } k = 1, \dots, N.$$

1.5. Review of related results. Theorem 1.2 generalizes the results of [35, Theorem 1.2(a), Theorem 1.3(a)], where stability was proved for systems of competing Lévy particles with (2) and (3), under the following conditions:

$$(9) \quad \nu_1 = \dots = \nu_N = \nu, \quad \int_{\mathbb{R}} z \nu(dz) = 0, \quad \int_{\mathbb{R}} |z| \nu(dz) < \frac{1}{N} \min(g_2 - g_1, \dots, g_N - g_{N-1}).$$

As the reader can see, these conditions (9) are much more restrictive than (8). Let us also mention our article [33], where we studied stability of systems of two competing Lévy particles, as well as explicit rate of exponential convergence of the gap process to its stationary distribution.

A related question is to find explicitly the stationary gap distribution π . For competing Brownian particles, under a so-called *skew-symmetry condition*, π has product of exponentials form, see [3, 27]. In other cases, an explicit form is not known. This stationary gap distribution satisfies a certain hard-to-solve integro-differential equation, which is called *basic adjoint relationship*, see [3, 38]. There are some tail estimates for π in [32]. However, for competing Lévy particles, the stationary gap distribution does not have a product form, see [12]. This makes finding an explicit form of π even more difficult than for competing Brownian particles.

One can also try to improve statements about convergence. For competing Brownian particles, if the system is stable, then the convergence of $Z(t)$ to π is actually exponentially fast: the TV distance at time t from π is estimated as $Ce^{-\varkappa t}$ for some positive constants C and \varkappa . This follows from the main result of [6]; see also [32, Proposition 4.1]. However, it is hard to find or estimate \varkappa . It was done only for $N = 2$ competing Brownian particles in [22]. See also related work [13] for systems of $N \geq 2$ competing Brownian particles, with $A = cI_N$ for $c > 0$ and I_N being the identity $N \times N$ matrix. For systems of $N = 2$ competing Lévy particles, such estimates were done in [33].

Finally, let us mention the papers [1, 21, 23, 12], which study stability for reflected diffusions with jumps.

1.6. Organization of the paper. Section 2 is devoted to some necessary background on related processes: reflected Brownian motion (RBM) with jumps on the half-line \mathbb{R}_+ and in the positive multidimensional orthant \mathbb{R}_+^d . In Sections 3 and 4, we prove Lemma 1.1 and Theorem 1.2, respectively.

2. BACKGROUND: REFLECTED BROWNIAN MOTION (RBM) WITH JUMPS

2.1. RBM with jumps on the half-line. Fix a *drift coefficient* μ , a *diffusion coefficient* σ^2 , and a *family of jump measures* $(\nu_x)_{x \in \mathbb{R}_+}$ such that every ν_x is a finite Borel measure on \mathbb{R}_+ . Assume this family is *weakly continuous*, in the sense that $\nu_y \Rightarrow \nu_x$ as $y \rightarrow x$. In addition, assume $\nu_x(\mathbb{R}_+) =: r > 0$ is independent of x . Let us define a Markov process $Z = (Z(t), t \geq 0)$ with state space \mathbb{R}_+ which:

- (a) behaves as an RBM on \mathbb{R}_+ with coefficients μ and σ^2 , except that:
- (b) it can jump, and the jump times form a Poisson process on \mathbb{R}_+ with intensity r , independently of the RBM from (a);
- (c) if it jumps from $x \in \mathbb{R}_+$, then the destination of the jump is distributed as $r^{-1}\nu_x(\cdot)$, independently of the jump time and the previous history of the process.

It is shown in [34] that such process exists in the weak sense and is unique in law, regardless of the initial condition; see also [33, Section 2]. It corresponds to a Feller continuous strong

Markov family with generator

$$\mathcal{L}f(x) := \mu f'(x) + \frac{1}{2}\sigma^2 f''(x) + \int_0^\infty [f(y) - f(x)] \nu_x(dy), \quad \text{for } f \in C^\infty(\mathbb{R}_+) \text{ with } f'(0) = 0.$$

Let $P^t(x, \cdot)$ be the transition kernel of this Markov family. Define the *effective drift coefficient*:

$$m(x) := \mu + \int_0^\infty [y - x] \nu_x(dy), \quad x \in \mathbb{R}_+.$$

The following statement is an analogue of [33, Corollary 3.4].

Lemma 2.1. *If $\overline{\lim}_{x \rightarrow \infty} m(x) < 0$, then there exists a unique stationary distribution π for this RBM with jumps. Moreover,*

$$\|P^t(x, \cdot) - \pi(\cdot)\|_{\text{TV}} \rightarrow 0 \quad \text{as } t \rightarrow \infty \text{ for every } x \rightarrow \infty.$$

Proof. Apply techniques from [33]: Take a C^∞ nondecreasing function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with

$$\varphi(x) = \begin{cases} 0, & x \leq 1, \\ x, & x \geq 2, \end{cases} \quad \varphi(x) \leq x \text{ for all } x \geq 0.$$

Then for $x \geq 2$, similarly to the proof of [33, Theorem 3.2], we get:

$$\mathcal{L}\varphi(x) = \mu + \int_0^\infty [\varphi(y) - x] \nu_x(dy) \leq m(x).$$

Therefore, $\overline{\lim}_{x \rightarrow \infty} \mathcal{L}\varphi(x) < 0$. Also, $\mathcal{L}\varphi$ is a continuous, hence bounded on compact intervals. Therefore, for some $b, k, s > 0$ we have:

$$(10) \quad \mathcal{L}\varphi(x) \leq -k + b1_{[0,s]}(x), \quad x \in \mathbb{R}_+.$$

This RBM with jumps has the following property: $P^t(x, C) > 0$ for all $t > 0$, $x \in \mathbb{R}_+$, and Borel subsets $C \subseteq \mathbb{R}_+$ with positive Lebesgue measure. This is noted in the proof of [33, Theorem 3.2]. Applying [24, Proposition 6.2.8], we get: any compact subset of \mathbb{R}_+ is petite for a skeleton Markov chain $(Z(n))_{n \geq 0}$. (We refer the reader to any standard reference, including [9, 25, 26, 24], for definitions of petite sets.) Using (10) and applying [26, Theorem 5.1], we complete the proof. \square

2.2. Semimartingale reflected Brownian motion (SRBM) in the orthant. Fix dimension $d \geq 2$. Let $S := \mathbb{R}_+^d$ be the positive d -dimensional orthant. First, let us define a *semimartingale reflected Brownian motion (SRBM) in the orthant S* , with drift vector $\mu \in \mathbb{R}^d$, a covariance matrix: a $d \times d$ -positive definite symmetric matrix $A = (a_{ij})_{1 \leq i, j \leq d}$, and a reflection matrix: a $d \times d$ -matrix $R = (r_{ij})_{1 \leq i, j \leq d}$ with $r_{ii} = 1$, $i = 1, \dots, d$. This process, denoted by $\text{SRBM}^d(R, \mu, A)$, is a Markov process with state space S which:

(a) behaves as a d -dimensional Brownian motion with drift vector μ and covariance matrix A in the interior of the orthant S ;

(b) on each face $F_i = \{x \in S \mid x_i = 0\}$ of the boundary ∂S , $i = 1, \dots, d$, the process is reflected in the direction of r_i , the i th column of R .

Remark 2. If $r_i = e_i$, the i th standard basis vector in \mathbb{R}^d , then the reflection is called *normal*; otherwise, it is called *oblique*.

Now, let us define this process formally.

Definition 3. Fix $x \in S$. Consider the following three processes:

- (a) a continuous adapted S -valued process $Z = (Z(t), t \geq 0)$;
- (b) a Brownian motion $B = (B(t), t \geq 0)$ in \mathbb{R}^d with drift vector μ and covariance matrix A , starting from $B(0) = x$;
- (c) another continuous adapted \mathbb{R}^d -valued process

$$L = (L(t), t \geq 0), \quad L(t) = (L_1(t), \dots, L_N(t)).$$

Assume that the following is true: $Z(t) = B(t) + RL(t)$ for $t \geq 0$, and for each $k = 1, \dots, d$, the process L_k is nondecreasing, $L_k(0) = 0$, and L_k can increase only when $Z_k = 0$. Then the process Z is called a *semimartingale reflected Brownian motion* (SRBM) in the orthant S , with drift vector μ , covariance matrix A , and reflection matrix R , starting from x .

The following existence and uniqueness result was proved in [28, 37].

Definition 4. A $d \times d$ -matrix $R = (r_{ij})_{1 \leq i, j \leq d}$ is called an \mathcal{S} -matrix if there exists a vector $u \in \mathbb{R}^d$, $u > 0$, such that $Ru > 0$. A *principal submatrix* of R is any submatrix of the form $(r_{ij})_{i, j \in I}$, where $I \subseteq \{1, \dots, d\}$ is a nonempty subset. A $d \times d$ -matrix R is called *completely- \mathcal{S}* if each of its principal submatrices is an \mathcal{S} -matrix.

Proposition 2.2. Fix any point $x \in S$. If R is a completely- \mathcal{S} matrix with unit elements on the main diagonal, then there exists in the weak sense a unique in law version of an $\text{SRBM}^d(R, \mu, A)$, starting from x . These processes for all $x \in S$ together form a Feller continuous strong Markov family.

2.3. SRBM with jumps in the orthant. We can augment an SRBM in the orthant with jumps. Assume, similarly to subsection 2.1, that we have a family of jump measures $(\nu_x)_{x \in S}$: every ν_x is a finite Borel measure on S , with $\nu_x(S) =: r > 0$ independent of x , and the family is *weakly continuous*: $\nu_y \Rightarrow \nu_x$ as $y \rightarrow x$. This process behaves as an $\text{SRBM}^d(R, \mu, A)$, except that it jumps with intensity r , and the destination of a jump from the point x is distributed as $r^{-1}\nu_x(\cdot)$. The following proposition can be proved similarly to its analogue from the subsection 2.2, see also [4, 34].

Proposition 2.3. This SRBM with jumps exists in a weak sense and is unique in law, for any initial condition. This is a Feller continuous strong Markov process. Its transition kernel has the following property: for $t > 0$, $x \in S$, and a Borel subset $C \subseteq S$ with positive Lebesgue measure, we have $P^t(x, A) > 0$.

As mentioned earlier, the papers [1, 21, 23] study stability for reflected diffusions with jumps.

3. PROOF OF LEMMA 1.1

Systems of competing Brownian particles (that is, without jumps, with $\Lambda = 0$), have weak existence and uniqueness in law; it follows from [5]. Now, let us construct a system of competing Lévy particles by *piecing out*. That is, we construct it as a continuous process until the first jump, then construct this jump; starting from the destination of the jump, we construct the second continuous piece of this process, until the second jump, etc. Take a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbf{P})$ with:

(a) for every $x \in \mathbb{R}^N$, infinitely many i.i.d. copies $X^{(x,n)} = (X_1^{(x,n)}, \dots, X_N^{(x,n)})$, $n = 1, 2, \dots$ of a system of N competing Brownian particles with drift vector g and covariance matrix A , starting from $X^{(x,n)}(0) = x$;

(b) infinitely many i.i.d. copies of an exponential random variable: $\eta_1, \eta_2, \dots \sim \text{Exp}(\lambda_0)$;

(c) infinitely many i.i.d. copies of an \mathbb{R}^N -valued random variable: $\zeta^{(1)}, \zeta^{(2)}, \dots \sim \lambda_0^{-1} \Lambda(\cdot)$.

We assume all these random objects are independent. Fix a starting point $x^{(0)} \in \mathbb{R}^N$, and let $X(0) = x^{(0)}$. Let $\tau_k := \eta_1 + \dots + \eta_k$, $k = 1, 2, \dots$; $\tau_0 := 0$. Define the system $X = (X(t), t \geq 0)$ on the time interval $(\tau_k, \tau_{k+1}]$ for each $k = 0, 1, \dots$, using induction by k . Assume it is already defined on the time interval $[0, \tau_k]$. For $t \in (\tau_k, \tau_{k+1})$, let $X(t) = X^{(x^{(k)}, k+1)}(t - \tau_k)$, where $x^{(k)} := X(\tau_k)$. Then let $\bar{x}^{(k+1)} = X^{(x^{(k)}, k+1)}(\eta_{k+1})$, and define

$$x^{(k+1)} = (x_1^{(k+1)}, \dots, x_N^{(k+1)}), \quad x_i^{(k+1)} := \bar{x}_i^{(k+1)} + \zeta_{q(i)}^{(k+1)}, \quad q := p_{x^{(k+1)}}^{-1}.$$

Next, let $X(\tau_{k+1}) := x^{(k+1)}$. Thus, we constructed the system X on time interval $(\tau_k, \tau_{k+1}]$. Since $\tau_k \rightarrow \infty$ a.s. as $k \rightarrow \infty$, this completes the construction of the system X .

4. PROOF OF THEOREM 1.2

Step 1. We can write (8) as

$$(11) \quad \frac{1}{k} (m_1 + \dots + m_k) > \frac{1}{N-k} (m_{k+1} + \dots + m_N), \quad k = 1, \dots, N-1.$$

The physical meaning of (11) is that if we split particles in two “clouds”, the bottom k ranked and the top $N-k$ ranked, then the average effective drift in the bottom “cloud” is strictly greater than the average effective drift in the top “cloud”. In turn, we can rewrite (11) as

$$(12) \quad \int_{\mathbb{R}^N} \Phi_k(v) \Lambda(dv) + G_k < 0, \quad k = 1, \dots, N-1,$$

$$(13) \quad \text{where } \Phi_k(v) := \frac{1}{N-k} \sum_{j=k+1}^N v_j - \frac{1}{k} \sum_{j=1}^k v_j, \quad \text{for } v = (v_1, \dots, v_N)' \in \mathbb{R}^N,$$

$$(14) \quad \text{and } G_k := \frac{1}{N-k} \sum_{j=k+1}^N g_j - \frac{1}{k} \sum_{j=1}^k g_j.$$

Step 2. The ranked particles Y_1, \dots, Y_N satisfy the following equation:

$$(15) \quad Y_k(t) = Y_k(0) + B_k(t) + \frac{1}{2} \ell_{(k-1, k)}(t) - \frac{1}{2} \ell_{(k, k+1)}(t) + J_k(t), \quad k = 1, \dots, N.$$

Here, we have:

(a) $B = (B_1, \dots, B_N)$ is an N -dimensional Brownian motion with drift vector g and covariance matrix A , starting from the origin;

(b) $(J_1(t), \dots, J_N(t)) = J(t)$, where $J = (J(t), t \geq 0)$ is a jump-only process that jumps with intensity λ_0 , independently of the Brownian motion B . If a jump starts from $y = (y_1, \dots, y_N) \in \mathbb{R}^N$, then the destination is given by $y + w$, where $w = (w_1, \dots, w_N)$, $w_i := v_{p_y^{-1}(i)}$, and $v = (v_1, \dots, v_N) \sim \lambda_0^{-1} \Lambda(\cdot)$ is independent of the history of the process J ;

(c) For each $k = 1, \dots, N-1$, the process $\ell_{(k,k+1)} = (\ell_{(k,k+1)}(t), t \geq 0)$ is a semimartingale local time of the k th gap Z_k at zero. This is a continuous nondecreasing process with $\ell_{(k,k+1)}(0) = 0$, which can increase only when $Z_k(t) = 0$, that is, $Y_k(t) = Y_{k+1}(t)$ (that is, the k th and $k+1$ st ranked particles collide). For simplicity of notation, we let $\ell_{(0,1)}(t) \equiv 0$ and $\ell_{(N,N+1)}(t) \equiv 0$.

The differential $d\ell_{(k,k+1)}(t)$ plays the role of push between ranked particles Y_k and Y_{k+1} : When they collide, they “want to cross each other” as Brownian motions would do. But they “are not allowed” to do this, because of the inequality $Y_k(t) \leq Y_{k+1}(t)$. The push $d\ell_{(k,k+1)}(t)$ is split evenly between these two particles: one-half of this pushes the particle Y_k down, and the other one-half pushes the particle Y_{k+1} up. This way the rankings $Y_k(t) \leq Y_{k+1}(t)$ are preserved.

This statement is already proved in [3] for the case of no jumps (that is, competing Brownian particles) and a diagonal matrix A . For the general covariance matrix A , the proof is completely analogous. The case with jumps reduces to the case without jumps as in [35].

Step 3. From Step 2, plugging into (4), one can show that the gap process is an SRBM in the orthant \mathbb{R}_+^{N-1} with drift vector

$$\mu = (g_2 - g_1, g_3 - g_2, \dots, g_N - g_{N-1}),$$

covariance matrix $\Sigma = (\sigma_{kl})$ with

$$\sigma_{kl} = a_{k+1,l+1} - a_{k,l+1} - a_{k+1,l} + a_{k,l}, \quad k, l = 1, \dots, N-1,$$

reflection matrix (which is tridiagonal):

$$(16) \quad R = \begin{bmatrix} 1 & -1/2 & 0 & 0 & \dots & 0 & 0 \\ -1/2 & 1 & -1/2 & 0 & \dots & 0 & 0 \\ 0 & -1/2 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 0 & \dots & 1 & -1/2 \\ 0 & 0 & 0 & 0 & \dots & -1/2 & 1 \end{bmatrix}$$

and with family $(\xi_z)_{z \in \mathbb{R}_+^{N-1}}$ of jump measures defined as follows. For every $z \in \mathbb{R}_+^{N-1}$, the measure ξ_z is a push-forward of the measure Λ under the mapping

$$\Theta_z : (v_1, \dots, v_N) \mapsto (y_{\mathbf{p}_y(2)} - y_{\mathbf{p}_y(1)}, \dots, y_{\mathbf{p}_y(N)} - y_{\mathbf{p}_y(N-1)}),$$

$$\text{where } y = (y_1, \dots, y_N), \quad y_i := \sum_{j=1}^{i-1} z_j + v_i, \quad i = 1, \dots, N.$$

It is possible to prove that the matrix R from (16) is, in fact, completely- \mathcal{S} ; see, for example, [19, Subsection 2.1]. In addition, the mapping $z \mapsto \Theta_z(v)$ is continuous for every $v \in \mathbb{R}^N$, and therefore the family $(\xi_z)_{z \in \mathbb{R}_+^{N-1}}$ is weakly continuous.

Step 4. Take a $k = 1, \dots, N-1$, and let us show that the collection of random variables $Z_k = (Z_k(t), t \geq 0)$ is tight in \mathbb{R}_+ . This constitutes the crux of the proof, done in Steps 5-8. Let e_1, \dots, e_N be the standard basis vectors in \mathbb{R}^N . Then $G_k = g \cdot u_k$, where

$$u_k := \frac{1}{N-k} \sum_{j=k+1}^N e_j - \frac{1}{k} \sum_{j=1}^k e_j \in \mathbb{R}^N, \quad k = 1, \dots, N.$$

Step 5. For $t \geq 0$, we have:

$$0 \leq Z_k(t) \equiv Y_{k+1}(t) - Y_k(t) \leq \frac{1}{N-k} \sum_{j=k+1}^N Y_j(t) - \frac{1}{k} \sum_{j=1}^k Y_j(t) =: V_k(t).$$

Split the particles into two groups: the bottom one, which contains particles with ranks $1, \dots, k$, and the top one, which contains particles with ranks $k+1, \dots, N$. We shall call these groups *clouds*. Assume that on a certain time interval, particles do not jump between the clouds. That is, no named particle X_i can jump and change ranks in a way that moves it from one cloud to the other one. If, in addition, $Z_k(t) > 0$ on this time interval, then the process V_k behaves as a reflected Brownian motion on \mathbb{R}_+ with drift coefficient $g \cdot u_k$, diffusion coefficient $\Sigma u_k \cdot u_k$, and family of jump measures $(\nu_x)_{x \geq 0}$. Here ν_x is defined as the push-forward of the measure Λ with respect to the mapping $F_x : v \mapsto x + \Phi_k(v)$, where Φ_k is taken from (13). Therefore, the process Z_k on this time interval is stochastically dominated by this reflected Brownian motion (RBM), even if we omit the condition $Z_k(t) > 0$. This is straightforward to prove by considering the cases $Z_k(t) > 0$ and $Z_k(t) = 0$. If the process Z_k was dominated by this RBM all of the time, then it is easy to prove tightness of $(Z_k(t), t \geq 0)$. Indeed, from (12) we get:

$$(17) \quad \int_{\mathbb{R}} [z - x] \nu_x(dz) + G_k = \int_{\mathbb{R}^N} [F_x(v) - x] \Lambda(dv) + G_k = \int_{\mathbb{R}^N} \Phi_k(v) \Lambda(dv) + g \cdot u_k < 0.$$

By Lemma 2.1, the family $Z_k = (Z_k(t), t \geq 0)$ would be tight. However, particles can jump from one cloud to the other. This is the main difficulty of this proof. Our idea is to note that, as $Z_k(t)$ becomes large, the clouds are far away from each other, and such exchange by jumps becomes very unlikely.

Step 6. Suppose t is the moment of the jump. Now, allow this undesirable event (exchanges between clouds) to happen. If the process V_k was at a point $x \in \mathbb{R}_+$ just before this jump, then the destination is governed by a certain probability measure q_x . However, if the displacement of the j th ranked particle during this jump is equal to v_j , then the destination of the jump satisfied $Z_k(t) \leq x + |v_1| + \dots + |v_N|$. Therefore, $q_x \preceq \bar{q}_x$, where \bar{q}_x is the push-forward of the measure $\lambda_0^{-1} \Lambda(\cdot)$ with respect to the mapping $H_x : v \mapsto x + \Psi(v)$, where $\Psi(v) := |v_1| + \dots + |v_N|$.

One idea is to dominate $Y_k(t)$ by an RBM on \mathbb{R}_+ with the same drift coefficient G_k and diffusion coefficient $\Sigma u_k \cdot u_k$, but with the family of jump measures $(\lambda_0 \bar{q}_x)_{x \geq 0}$. However, then we would not obtain tightness, at least not under conditions of Theorem 1.2. In the paper [35], they take this approach and therefore are forced to impose much more restrictive conditions in place of (8). However, this undesirable event (jumping of particles between clouds) is unlikely to happen if $Z_k(t)$ is large. We can bypass this obstacle by considering a dominating family of jump measures $(\bar{\nu}_x)_{x \geq 0}$, which, loosely speaking, incorporates two cases:

- (a) if there is no exchange of particles between clouds, then $\bar{\nu}_x = \nu_x$;
- (b) if there is such an exchange, then $\bar{\nu}_x$ is the push-forward of Λ with respect to H_x .

This worst case can happen only if $v_j - v_i \leq -x$ for some $i = 1, \dots, k$ and $j = k+1, \dots, N$. Denote the set of such $v = (v_1, \dots, v_N) \in \mathbb{R}^N$ by $E(x)$. Let $\bar{\nu}_x$ be the push-forward of Λ with respect to the mapping

$$\bar{F}_x : v \mapsto \begin{cases} H_x(v), & v \in E(x); \\ F_x(v), & v \notin E(x). \end{cases}$$

Step 7. However, we still cannot claim that the process Z_k is dominated by an RBM on \mathbb{R}_+ with drift coefficient G_k , diffusion coefficient $u'_k A u_k$, and the family of jump measures $(\bar{\nu}_x)_{x \geq 0}$.

The reason is that this family of measures is not stochastically ordered: that is, it is not true that $\bar{\nu}_{x'} \preceq \bar{\nu}_{x''}$ if $x' \leq x''$. This, in turn, is because it is not necessarily true that $H_{x'}(v) \leq H_{x''}(v)$ if $x' \leq x''$. Stochastic ordering of the dominating family of jump measures is crucial, see [33, Section 4] for a very similar setting. Thus, we must again modify the mapping \bar{F}_x . Let us state and prove a simple technical lemma.

Lemma 4.1. *As $x \rightarrow \infty$, we have:*

$$\int_{\mathbb{R}^N} [\bar{F}_x(v) - x] \Lambda(dv) \rightarrow \int_{\mathbb{R}^N} \Phi_k(v) \Lambda(dv).$$

Proof. Observe that $\Lambda(E(x)) \rightarrow 0$ as $x \rightarrow \infty$, and $F_x(v) = \bar{F}_x(v)$ for $v \in \mathbb{R}^N \setminus E(x)$. Also, for all $v \in \mathbb{R}^N$ and $x \in \mathbb{R}_+$, we have:

$$|\bar{F}_x(v) - x| \leq \Psi(v), \quad \text{and} \quad \int_{\mathbb{R}^N} \Psi(v) \Lambda(dv) < \infty.$$

Therefore, we apply Lebesgue dominated convergence theorem and note that for all $x \in \mathbb{R}_+$,

$$\int_{\mathbb{R}^N} (F_x(v) - x) \Lambda(dv) = \int_{\mathbb{R}^N} \Phi_k(v) \Lambda(dv).$$

□

The reason that $H_x(v)$ is not necessarily increasing with respect to x is that the “worst case domain” $E(x)$ (where \bar{F}_x becomes really large) is decreasing as x increases. This suggests a solution: cut the RBM at a certain level $c > 0$. That is, define it on $[c, \infty)$ instead of \mathbb{R}_+ . Implement the “worst case mapping” H_x on the domain $E(c)$ instead of $E(x)$, for $x \geq c$. If the process “wants” to jump below c , redirect it instead to c . Take c large enough so that

$$(18) \quad \int_{\mathbb{R}^N} [\bar{F}_c(v) - c] \Lambda(dv) + G_k < 0.$$

That such c exists follows from Lemma 4.1 and (17).

Step 8. Now, consider a new RBM: $U_k = (U_k(t), t \geq 0)$ on $[c, \infty)$ instead of \mathbb{R}_+ , with the same drift and diffusion coefficients: G_k and $\Sigma u_k \cdot u_k$, and with family of jump measures $(\mu_x)_{x \geq c}$, defined as push-forwards of Λ with respect to the following mapping:

$$\tilde{F}_x : v = (v_1, \dots, v_N)' \mapsto \begin{cases} H_x(v) = x + \Psi(v), & \text{if } v \in E(c); \\ F_x(v) \vee c, & \text{if } v \notin E(c). \end{cases}$$

Then $\tilde{F}_{x'}(v) \leq \tilde{F}_{x''}(v)$ if $x' \leq x''$, and so this family of measures is stochastically ordered. Now, let us show an analogue of Lemma 4.1.

Lemma 4.2. *We have:*

$$\lim_{x \rightarrow \infty} \int_{\mathbb{R}^N} [\tilde{F}_x(v) - x] \Lambda(dv) + G_k < 0.$$

Proof. Let $D(x) := \{v \in \mathbb{R}^N \mid x + \Phi_k(v) < c\}$. Then $D(x) \downarrow \emptyset$ as $x \rightarrow \infty$, and so $\Lambda(D(x)) \rightarrow 0$. However, if $v \in \mathbb{R}^N \setminus D(x)$, then $\tilde{F}_x(v) - x = \bar{F}_c(v) - c$. Similarly to the proof of Lemma 4.1, we argue that

$$\lim_{x \rightarrow \infty} \int_{\mathbb{R}^N} [\tilde{F}_x(v) - x] \Lambda(dv) = \int_{\mathbb{R}^N} [\bar{F}_c(v) - c] \Lambda(dv).$$

The rest follows from (18). □

It is not difficult to prove that Z_k is dominated by U_k : we consider the interval of continuity before the first jump, then this jump, then the next interval of continuity, the second jump, etc. From the results of Appendix, the family of random variables $(U_k(t), t \geq 0)$ is tight. Therefore, the same is true for $(Z_k(t), t \geq 0)$.

Step 9. Now we know that for each $k = 1, \dots, N - 1$, the family of random variables $(Z_k(t), t \geq 0)$ is tight. Therefore, the family $(Z(t), t \geq 0)$ of \mathbb{R}_+^{N-1} -valued random variables is tight. From Step 3 of this proof and Proposition 2.3, the gap process is a Feller continuous strong Markov process. It also has the property $P^t(z, C) > 0$, for every $t > 0$, $z \in \mathbb{R}_+^{N-1}$, and any Borel subset $C \subseteq \mathbb{R}_+^{N-1}$ with positive Lebesgue measure. Following the proof of [35, Theorem 1.2], parts (a), sub-parts (3) and (4), we complete the proof of Theorem 1.2.

ACKNOWLEDGEMENTS

This research was partially supported by NSF grants DMS 1409434 and DMS 1405210. The author is grateful to RICARDO FERNHOLZ, SOUMIK PAL, and MYKHAYLO SHKOLNIKOV for advice and discussion.

REFERENCES

- [1] RAMI ATAR, AMARJIT BUDHIRAJA (2002). Stability Properties of Constrained Jump-Diffusion Processes. *Electr. J. Probab.* **7** (22), 1-31.
- [2] ADRIAN D. BANNER, E. ROBERT FERNHOLZ, IOANNIS KARATZAS (2005) Atlas Models of Equity Markets. *Ann. Appl. Probab.* **15** (4), 2996-2330.
- [3] ADRIAN D. BANNER, E. ROBERT FERNHOLZ, TOMOYUKI ICHIBA, IOANNIS KARATZAS, VASSILIOS PAPAATHANAKOS (2011). Hybrid Atlas Models. *Ann. Appl. Probab.* **21** (2), 609-644.
- [4] RICHARD F. BASS (1979). Adding and Subtracting Jumps from Markov Processes. *Trans. Amer. Math. Soc.* **255**, 363-376.
- [5] RICHARD F. BASS, ETIENNE PARDOUX (1987). Uniqueness for Diffusions with Piecewise Constant Coefficients. *Probab. Th. Rel. Fields* **76** (4), 557-572.
- [6] AMARJIT BUDHIRAJA, CHIHOOON LEE (2007). Long Time Asymptotics for Constrained Diffusions in Polyhedral Domains.
- [7] SOURAV CHATTERJEE, SOUMIK PAL (2010). A Phase Transition Behavior for Brownian Motions Interacting Through Their Ranks. *Probab. Th. Rel. Fields* **147** (1), 123-159.
- [8] AMIR DEMBO, MYKHAYLO SHKOLNIKOV, S. R. SRINIVASA VARADHAN, OFER ZEITOUNI. Large Deviations for Diffusions Interacting Through Their Ranks. *Comm. Pure Appl. Math.* **69** (7), 1259-1313.
- [9] DOUGLAS DOWN, SEAN P. MEYN, RICHARD L. TWEEDIE (1995). Exponential and Uniform Ergodicity of Markov Processes. *Ann. Probab.* **23** (4), 1671-1691.
- [10] E. ROBERT FERNHOLZ (2002). *Stochastic Portfolio Theory*. Applications of Mathematics **48**. Springer.
- [11] E. ROBERT FERNHOLZ, IOANNIS KARATZAS (2009) Stochastic Portfolio Theory: An Overview. *Handbook of Numerical Analysis: Mathematical Modeling and Numerical Methods in Finance*, 89-168. Elsevier.
- [12] FABRICE M. GUILLEMIN, RAVI R. MAZUMDAR, FRANCISCO J. PIERA (2008). On Product-Form Stationary Distributions for Reflected Diffusions with Jumps in the Positive Orthant. *Adv. Appl. Probab.* **37** (1), 212-228.
- [13] TOMOYUKI ICHIBA, SOUMIK PAL, MYKHAYLO SHKOLNIKOV (2013). Convergence Rates for Rank-Based Models with Applications to Portfolio Theory. *Probab. Th. Rel. Fields* **156**, 415-448.
- [14] TOMOYUKI ICHIBA, IOANNIS KARATZAS, MYKHAYLO SHKOLNIKOV (2013). Strong Solutions of Stochastic Equations with Rank-Based Coefficients. *Probab. Th. Rel. Fields* **156**, 229-248.
- [15] BENJAMIN JOURDAIN, FLORENT MALRIEU (2008). Propagation of Chaos and Poincare Inequalities for a System of Particles Interacting Through Their cdf. *Ann. Appl. Probab.* **18** (5), 1706-1736.
- [16] BENJAMIN JOURDAIN, JULIEN REYGNIER (2013). Propagation of Chaos for Rank-Based Interacting Diffusions and Long-Time Behaviour of a Scalar Quasilinear Parabolic Equation. *SPDE Anal. Comp.* **1** (3), 455-506.

- [17] BENJAMIN JOURDAIN, JULIEN REYGNER (2014). The Small Noise Limit of Order-Based Diffusion Processes. *Electr. J. Probab.* **19** (29), 1-36.
- [18] BENJAMIN JOURDAIN, JULIEN REYGNER (2015). Capital Distribution and Portfolio Performance in the Mean-Field Atlas Model. *Ann. Finance* **11** (2), 151-198.
- [19] IOANNIS KARATZAS, SOUMIK PAL, MYKHAYLO SHKOLNIKOV (2016). Systems of Brownian Particles with Asymmetric Collisions. *Ann. Inst. H. Poincaré* **52** (1), 323-354.
- [20] IOANNIS KARATZAS, ANDREY SARANTSEV (2016). Diverse Market Models of Competing Brownian Particles with Splits and Mergers. *Ann. Appl. Probab.* **26** (3), 1329-1361.
- [21] OFFER KELLA, WARD WHITT (1996). Stability and Structural Properties of Stochastic Storage Networks. *J. Appl. Probab.* **33** (4), 1169-1180.
- [22] ROBERT B. LUND, SEAN P. MEYN, RICHARD L. TWEEDIE (1996). Computable Exponential Convergence Rates for Stochastically Ordered Markov Processes. *Ann. Appl. Probab.* **6** (1), 218-237.
- [23] RAVI R. MAZUMDAR, FRANCISCO J. PIERA (2008). Comparison Results for Reflected Jump-Diffusions in the Orthant with Variable Reflection Directions and Stability Applications. *Electr. J. Probab.* **13** (61), 1886-1908.
- [24] SEAN P. MEYN, RICHARD L. TWEEDIE (2009). *Markov Chains and Stochastic Stability*. Cambridge University Press.
- [25] SEAN P. MEYN, RICHARD L. TWEEDIE (1993). Stability of Markovian Processes II: Continuous-Time Processes and Sampled Chains. *Adv. Appl. Probab.* **25** (3), 487-517.
- [26] SEAN P. MEYN, RICHARD L. TWEEDIE (1993). Stability of Markovian Processes III: Foster-Lyapunov Criteria for Continuous-Time Processes. *Adv. Appl. Probab.* **25** (3), 518-548.
- [27] SOUMIK PAL, JIM PITMAN (2008). One-Dimensional Brownian Particle Systems with Rank-Dependent Drifts. *Ann. Appl. Probab.* **18** (6), 2179-2207.
- [28] I. MARTIN REIMAN, RUTH J. WILLIAMS (1988). A Boundary Property of Semimartingale Reflecting Brownian Motions. *Probab. Th. Rel. Fields* **77** (1), 87-97.
- [29] JULIEN REYGNER (2015). Chaoticity of the Stationary Distribution of Rank-Based Interacting Diffusions. *Electr. Comm. Probab.* **20** (60), 1-20.
- [30] ANDREY SARANTSEV (2015). Comparison Techniques for Competing Brownian Particles. To appear in *J. Th. Probab.* Available at arXiv:1305.1653.
- [31] ANDREY SARANTSEV (2015). Triple and Simultaneous Collisions of Competing Brownian Particles. *Electr. J. Probab.* **20** (29), 1-28.
- [32] ANDREY SARANTSEV (2016). Reflected Brownian Motion in a Convex Polyhedral Cone: Tail Estimates for the Stationary Distribution. To appear in *J. Th. Probab.* Available at arXiv:1509.01781.
- [33] ANDREY SARANTSEV (2016). Explicit Rates of Exponential Convergence for Reflected Jump-Diffusions on the Half-Line. Available at arXiv:1509.01783.
- [34] STANLEY A. SAWYER (1970). A Formula for Semigroups, with an Application to Branching Diffusion Processes. *Trans. Amer. Math. Soc.* **152** (1), 1-38.
- [35] MYKHAYLO SHKOLNIKOV (2011). Competing Particle Systems Evolving by Interacting Lévy Processes. *Ann. Appl. Probab.* **21** (5), 1911-1932.
- [36] MYKHAYLO SHKOLNIKOV (2012). Large Systems of Diffusions Interacting Through Their Ranks. *Stoch. Proc. Appl.* **122** (4), 1730-1747.
- [37] LISA M. TAYLOR, RUTH J. WILLIAMS (1993). Existence and Uniqueness of Semimartingale Reflecting Brownian motions in an Orthant. *Probab. Th. Rel. Fields* **96** (3), 283-317.
- [38] RUTH J. WILLIAMS (1995). Semimartingale Reflecting Brownian Motions in the Orthant. *Stochastic networks*, IMA Vol. Math. Appl. **71**, 125-137. Springer.

DEPARTMENT OF STATISTICS AND APPLIED PROBABILITY, UNIVERSITY OF CALIFORNIA, SANTA BARBARA
 E-mail address: sarantsev@pstat.ucsb.edu